J. Fluid Mech. (1985), *vol.* 151, *pp.* 189–217
Printed in Great Britain

Instability and wave over-reflection in stably stratified shear flow

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(Received 29 December 1983 and in revised form 31 July 1984)

We reexamine the related problems of instability of parallel shear flows and over-reflection of internal waves at a critical level, concentrating on the stratified case. Our primary aim is to delineate the specific aspects of a flow that permit overreflection and instability. A related and partly realized aim is to develop a mechanistic 'picture' of how over-reflection and instability work. In the course of this study we have also uncovered some new results concerning the instability of stratified shear flows – showing how regions of enhanced static stability and enhanced damping can destabilize otherwise stable flows.

For the scattering of steady plane waves, we show that, of the conditions found by Lindzen & Tung (1978), in the unstratified case, only the existence of wave-propagation regions above and below the critical level is always necessary for over-reflection (at least in the absence of damping), although a trapping region around the critical level and a reflecting surface bounding the upper wave region may play crucial roles in some cases. Our results suggest that the role of the upper wave region may be to allow a wave flux through the critical level. Moreover, we show numerically that the effect of an upper wave region can be mimicked by a region of localized damping which leads to over-reflection as well.

We also consider an initial-value problem, using numerical methods. When a wave is incident on the incident level, the reflection and transmission coefficients grow smoothly to their final values. The rate of growth depends on the flow parameters, but there is some evidence to suggest there is a characteristic timescale involved that depends only on the shear (and not on wave travel time). This fits a mechanistic picture of over-reflection and instability that we describe, in which the essential part is a kinematic interaction between wave and mean flow at the critical level, depending only on shear.

1. Introduction

In this paper we shall discuss the linear stability of a stratified shear flow and the reflection and transmission of internal waves at a critical level with the aims of delineating the conditions under which instability, over-reflection and overtransmission can occur. We find that the most meaningful expression of these conditions is in terms of the wave-propagation properties of the basic state being perturbed. An additional, but less clearcut, aim is to develop a mechanistic picture of how instability and over-reflection function. In passing we will present some new results on the instability of stratified shear flows – showing how otherwise-stable flows can be *destabilized* by introducing regions of *enhanced* Richardson number and/or damping.

Recent work by Lindzen and others (Lindzen 1974; Lindzen, Farrell & Tung 1980; Lindzen & Rosenthal 1976, 1981, 1983; Lindzen & Tung 1978; Rosenthal & Lindzen 1983*a*, *b*) has shown instability and over-reflection to be closely related. Their results show that instability may be viewed as an interaction between internal waves and the mean flow. This interaction occurs at a critical level (where the horizontal phase speed of the wave is equal to the mean shear velocity $U(z)$), where the internal waves can be over-reflected. It has been shown that instability results when over-reflected waves are contained in regions of appropriate dimensions. We discuss this in detail in §3.

It is known that over-reflection can occur at a critical level only if the Richardson number there is less than a quarter (Miles 1961; Howard 1961), but this alone is not a sufficient condition. Having considered a number of different problems, stratified and unstratified, with and without rotation and baroclinic effects, Lindzen & Tung (1978) proposed four conditions that appeared to be necessary for over-reflection. These conditions are phrased in terms of the 'wave geometry' of the problem; that is, the particular configuration of 'wave regions', where the internal wave may propagate normally to the basic shear flow, and 'trapping regions', where the solution will show exponential growth or decay. In this paper we confine our attention mainly to the stratified problem, because it offers the flexibility to specify this geometry, unlike the unstratified shear-flow problem, where the geometry is largely fixed by the existence of an inflection point.

In §4 we present exact solutions in some simple cases which show that two of these conditions, a 'trapping region' near the critical level, and a reflecting surface to one side of it, are not in fact necessary, though each may have a role to play in some cases.

The remaining conditions require that there should be a 'wave region' on each side of the critical lavel. Clearly, one such region is necessary, for there must be waves somewhere if there is to be any reflection at all. For inviscid problems, the second region also proves to be essential. This immediately accounts for the stability of stratified shear flow where shear $d\bar{u}/dz$ and static stability are constant and where $Ri < \frac{1}{4}$; namely, such a state supports no wave propagation. The stability of this profile was shown using conventional methods by Taylor (1931), Goldstein (1931) and Case (1960). The problem is sometimes referred to as the Taylor–Goldstein problem. Two regions of wave propagation can be introduced into the problem by increasing the static stability in two distinct regions sufficiently to raise Ri, the Richardson number, above $\frac{1}{4}$ in these two regions. This is shown to produce both over-reflection and instability.

In §5 we suggest a role for the second wave region, and present some evidence to support it. Our suggestion is that the role of the second wave region may be to allow a wave flux to pass through the critical level. It seems plausible that, without such a flux, the wave may not 'see', or properly interact with, the critical level; if there is no second wave region, and the fluid is bounded by a solid wall or extends to infinity, then the Eliassen–Palm theorems state that there is no such flux.

If this suggestion is correct, then anything that induces a wave flux through the critical level should give rise to over-reflection. To test this, we introduced a region of linear damping above the critical level in a case where the second wave region was absent (the problem is otherwise inviscid). The damping region acts as an energy sink, and thus draws a wave flux through the critical level from the wave region below it. We find that over-reflection can indeed be induced by the damping, and so feel that this is evidence in favour of our suggestion. It also provides a second way to destabilize the Taylor–Goldstein profile.

This effect of localized damping also suggests an explanation in terms of over-reflection for the instability of viscous Poiseuille flow, where the inviscid problem is stable. It seems plausible that the viscosity, which acts mostly near the boundaries, will exhibit a local character, and may therefore be able to induce an over-reflection not possible in the inviscid problem. Lindzen and Rambaldi have confirmed this numerically and the results will be presented separately.

Our suggestion that existence of a wave flux away from the critical level is necessary for over-reflection is consistent with the results of Lin & Lau (1979) and others concerning over-reflection and instability in galaxies. It is also reminiscent of the argument of Bretherton (1966*a*) for the existence of critical-level instability in a baroclinic flow.

In §6 we examine how the reflection coefficient develops in an initial-value problem. Several examples are studied numerically. The reflection (and transmission) coefficient grows smoothly to its final value, much of the growth being linear in time. The rate of growth and the time taken to reach a steady value vary from case to case, and we deduce an empirical dependence on the basic-state parameters. The existence of a characteristic timescale, independent of Richardson number, is suggested by the results. This is *not* a travel time in any bounded wave region; such a time may be relevant in certain problems, but it will always enter as a *second*, independent timescale. We give an example to show that estimates of growth rates of instabilities resulting from over-reflection can be substantially improved by allowing for the time development of the reflection coefficient. This allows the unambiguous identification of instability with over-reflected waves.

The existence of a characteristic timescale, depending only on the shear, is consistent with the first part of a mechanistic picture for over-reflection and instability which we describe in §7. The picture consists of three parts: first, the wave/mean-flow interaction at the critical level, which is essentially kinematic, depending only on shear; secondly, the maintenance of a disturbance at the critical level, which requires the conditions discussed in §§3–5; thirdly, for a normal-mode instability, certain extra conditions akin to quantization, of a technical rather than basic nature, are required (see Lindzen & Rosenthal 1976; Lindzen, Farrell & Tung 1980). Such theorems as the Rayleigh inflection-point theorem, the Fjørtoft theorem and the Miles–Howard theorem are primarily concerned with the conditions needed for a wave to reach a critical level.

2. Necessary conditions for instability

We consider an inviscid incompressible fluid in motion in a vertical plane with horizontal coordinate x and vertical coordinate z, and are interested in the stability, or otherwise, of a basic state in which the fluid has density $\rho_0(z)$ and velocity $(U(z), 0)$. This state is assumed to be statistically stable:

$$N^2(z) = -\frac{g}{\rho_0}\frac{\mathrm{d}\rho_0}{\mathrm{d}z} > 0. \tag{2.1}$$

If (u, w), p and ρ' are small-amplitude perturbations in the velocity, pressure and density fields, they satisfy the linearized equations of motion

$$\left.\begin{aligned} u_t+Uu_x+wU_z &= -\frac{1}{\rho_0}p_x, \\ w_t+Uw_x &= -\frac{1}{\rho_0}p_z-\frac{1}{\rho_0}\rho', \\ \rho'_t+U\rho'_x+w\rho_{0z} &= 0, \\ u_x+w_z &= 0. \end{aligned}\right\} \tag{2.2}$$

Introducing a stream function ψ, the vorticity ω and a scaled density ρ given by

$$\omega = w_x-u_z, \quad w = -\psi_x, \quad u = \psi_z, \quad \rho = \frac{g}{\rho_0}\rho',$$

and making the Boussinesq approximation, ρ_0 = constant in the first two equations of (2.2), the equations become

$$\omega_t+U\omega_x+U_{zz}\psi_x = -\rho_x, \quad \rho_t+U\rho_x+N^2\psi_x = 0, \quad \psi_{xx}+\psi_{zz} = -\omega. \tag{2.3}$$

If 'normal-mode' solutions are sought in which

$$\psi(x,z,t) = \Psi(z)\,e^{ik(x-ct)}, \tag{2.4}$$

and ρ, ω are similarly expressed, the equations reduce to

$$\Psi_{zz}+Q(z)\,\Psi = 0, \tag{2.5}$$

where the index of refraction Q is

$$Q(z) = -k^2-\frac{U_{zz}}{U-c}+\frac{N^2}{(U-c)^2}. \tag{2.6}$$

For boundary conditions we shall want to consider several possibilities. If the fluid is bounded above by a solid wall at z_T then the vertical velocity should vanish there, or

$$\Psi(z_T) = 0. \tag{2.7a}$$

If the fluid is unbounded above we shall assume $Q(z)\rightarrow Q_\infty$ = constant as $z\rightarrow\infty$. Then the appropriate boundary condition is

$$\Psi(z)\rightarrow 0 \quad \text{as } z\rightarrow\infty \tag{2.7b}$$

if $Q_\infty < 0$, or a radiation condition

$$\Psi_z-iQ_\infty^{\frac{1}{2}}\,\Psi\rightarrow 0 \quad \text{as } z\rightarrow\infty \tag{2.7c}$$

if $Q_\infty > 0$, where the sign of $Q_\infty^{\frac{1}{2}}$ is chosen to be the same as that of $[U-c]_{z\rightarrow\infty}$.

If a similar condition is applied at some lower boundary, one obtains an eigenvalue problem for c, the stability problem. The basic flow is unstable if there is an eigenvalue with positive imaginary part. If no lower boundary condition is applied, c may be specified in advance, and one obtains the scattering problem. There will always be a solution, unique up to a constant factor.

In the stability problem, necessary conditions for instability may be derived by multiplying (2.5) by $(u-c)^n\,\Psi^*$, for various n, and integrating over the fluid (Howard 1961). The resulting conditions have two major drawbacks. First, because of their global nature, they give no insight into the physical mechanism responsible for any instability. Secondly, they involve the modal solution Ψ and its phase speed c. The only condition that can be applied to the basic state alone is that the local Richardson number

$$Ri(z) = \frac{N^2(z)}{(\mathrm{d}U/\mathrm{d}z)^2} \tag{2.8}$$

must fall below $\frac{1}{4}$ at some point within the fluid for instability to be possible.

The other conditions that can be derived in this way are the semicircle theorem (Howard 1961), which requires that the complex phase speed c of an unstable model must lie within the semicircle in the upper half of the complex plane which has the range of U for its diameter, and a generalized version of Rayleigh's theorem (Synge 1933), which requires that

$$\frac{\mathrm{d}^2U}{\mathrm{d}z^2} - \frac{2N^2(U-c_r)}{|U-c|^2} \tag{2.9}$$

must change sign within the fluid. Unfortunately, this is of little use, as it is satisfied by almost all modes that have c_r within the range of U.

Consider for a moment the unstratified case $N^2(z) \equiv 0$. The Richardson-number criterion is no longer relevant (it is always satisfied). Instead there is Fjørtoft's criterion (Fjørtoft 1950), which requires that U and U_{zz} must be negatively correlated in the sense that

$$\int_{z_B}^{z_T} UU_{zz} \left| \frac{\Psi}{U-c} \right|^2 \mathrm{d}z < 0 \tag{2.10}$$

for any unstable mode. More importantly however, (2.9) reduces to Rayleigh's theorem (Rayleigh 1880), which requires that U_{zz} change sign within the fluid. Thus the relevant criterion for instability in the unstratified case is very different from that when $N^2 \neq 0$. This reflects the singularity of (2.5) in the limit $N^2 \to 0$, in which the order of the pole in Q at $U = c$ is reduced.

Rayleigh's criterion may be derived in other ways that are more physically motivated. Taylor (1915) showed that, in a growing perturbation, the total momentum in a layer of fluid will increase or decrease according to the sign of U_{zz}, and, since the total momentum of an inviscid fluid is fixed, such perturbations are possible only if U_{zz} changes sign. Taylor used the fact that individual fluid particles retain their initial vorticity throughout any motion to relate the momentum changes to the particle displacements during the disturbance. If $N^2 \neq 0$, however, buoyancy forces do change particle vorticity. It is possible to bring in particle displacements by using the fact that each particle preserves its density, but no useful results appear to come out of this approach. Lin (1955) argued that, since a vorticity gradient will support waves in the fluid, perturbations in regions of non-zero U_{zz} will be oscillatory in nature, and so instability will be possible only if U_{zz} vanishes at some point. Again this argument does not appear to have an analogue in the stratified case. In any event, neither of these arguments throws light explicitly on the physical mechanisms responsible for instability.

Bretherton (1966*a*) showed that in the baroclinic-instability problem (which is mathematically the same as unstratified shear flow), a neutral normal-mode solution that does not vanish at the critical level will have a downgradient flux of potential vorticity at the critical level which cannot be balanced anywhere else in the fluid unless the solution is in fact growing. This suggests that the critical level drives the instability. No analogue of this result, which was derived using the conservation of potential vorticity by a quasi-geostrophic flow, is yet available for stratified shear flow.

3. Wave over-reflection and instability

In the stratified problem, internal waves can be supported both by the density gradient and by any vorticity gradient. In fact, solutions of (2.5) will show oscillatory or exponential variation with z depending on whether Q is positive or negative, provided that Q does not vary too rapidly. As pointed out by Lindzen *et al.* (1980), solutions of (2.5) will be exponential near the critical level if $Ri = \text{constant} < \frac{1}{4}$, even though $Q > 0$.

Near any point where the solution is oscillatory, it may be approximated by the WKB solution

$$\begin{aligned}\Psi(z) &= AQ^{-\frac{1}{4}}(z)\exp\left\{\mathrm{i}\int^{z} Q^{\frac{1}{2}}(z')\,\mathrm{d}z'\right\} + BQ^{-\frac{1}{4}}(z)\exp\left\{-\mathrm{i}\int^{z} Q^{\frac{1}{2}}(z')\,\mathrm{d}z'\right\}\\ &= \Psi_{\mathrm{U}}(z) + \Psi_{\mathrm{D}}(z) \quad \text{(say)}.\end{aligned} \tag{3.1}$$

How to choose the correct branch of $Q^{\frac{1}{2}}$ in the vicinity of the critical level is discussed in detail by Booker & Bretherton (1967), but the solution (3.1) is in any case not valid near the critical level if $Ri < \frac{1}{4}$ there. We may thus take $Q^{\frac{1}{2}}$ to have the same sign as $U-c$.

With this choice, one finds (following Booker & Bretherton):

$$\begin{aligned}\overline{pw} &= -\rho_0(U-c)\,\overline{uw} = -\tfrac{1}{4}\mathrm{i}\rho_0 k(U-c)\,(\Psi^*\Psi_z - \Psi\Psi_z^*)\\ &= \tfrac{1}{2}\rho_0 k\,|U-c|\,\{|A|^2 - |B|^2\}.\end{aligned} \tag{3.2}$$

Since the contribution from Ψ_{U} is positive, and that from Ψ_{D} negative, the former represents a wave propagating upwards, the latter one downwards.

Suppose now the fluid is unbounded below, and there is some height z_{s} such that $Q(z)$ is slowly varying and positive ($Q(z) \geqslant \epsilon > 0$) for all $z < z_{\mathrm{s}}$, and there is no critical level in $z < z_{\mathrm{s}}$. The solution in this region is then approximately (3.1). With any of the upper boundary conditions (2.7), there is no source of waves above z_{s}. No lower boundary condition is needed in this scattering problem, so in view of (3.2),

$$R = \left|\frac{\Psi_{\mathrm{D}}}{\Psi_{\mathrm{U}}}\right| = \left|\frac{B}{A}\right| \tag{3.3}$$

is the reflection coefficient of the region of fluid above z_{s} (in the WKB approximation; see Appendix C for further discussion).

What may be said concerning the value of R? The Eliassen–Palm (1961) theorems state that

$$\overline{\rho w} = -\rho_0(U-c)\,\overline{uw} \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}z}(\overline{uw}) = 0, \tag{3.4}$$

except at a critical level, where there may be a jump in $\overline{uw}$. Thus, in the absence of a change of sign of $U-c$ in $z > z_{\mathrm{s}}$, $\overline{uw}$ will have the same value at $z = z_{\mathrm{s}}$ as at the upper boundary, and $\overline{\rho w}$ will have the same sign. Since at the upper boundary, $\overline{\rho w} \geqslant 0$, it follows from (3.2) and (3.3) that $R < 1$. If $U-c$ does change sign, at $z = z_{\mathrm{c}}$ say, things may be different. If $Ri(z_{\mathrm{c}}) > \frac{1}{4}$ then Booker & Bretherton (1967) showed that both upward- and downward-moving waves are effectively absorbed by the critical level, so again $R \leqslant 1$. If $Ri(z_{\mathrm{c}}) < \frac{1}{4}$, however, it is possible to have $R > 1$. This was first demonstrated numerically by Jones (1968).

Next suppose there is a wave that is over-reflected ($R > 1$). Suppose a solid boundary is placed at $z = z_{\mathrm{B}} < z_{\mathrm{s}}$, and such an upward-moving wave is excited near this lower boundary. Neglecting transient effects, the wave will be reflected down from above z_s with increased amplitude. On reaching z_B, it will be reflected back upwards with no change in amplitude. So long as this wave does not interfere destructively with the original wave, this bouncing process can continue, the amplitude of the wave increasing indefinitely, and an instability can result. This possibility was first pointed out by Lindzen (1974). The growth rate of the instability may be crudely estimated as

$$kc_i = \frac{1}{2\tau} \ln R, \tag{3.5}$$

where 2τ is the time taken for the wave to propagate from z_B to the reflecting surface and back.

Lindzen & Rosenthal (1976) showed that all gravity-wave instabilities of a Helmholtz velocity profile in a stratified fluid with a lower boundary could be identified with neutral over-reflected modes of the same fluid with no lower boundary. The growth rate given by (3.5) was always an overestimate of the actual growth rate, but this can be ascribed to the fact that transient effects have been neglected. We examine this in §6, and show that a much more accurate estimate can be made by allowing for the time the reflection coefficient takes to grow.

Lindzen and Rosenthal (Rosenthal & Lindzen 1983*a*,*b*; Lindzen & Rosenthal 1983) extended these results to a continuous profile, and also identified the traditional Kelvin–Helmholtz instability with such a mechanism. Lindzen & Tung (1978) showed that the same ideas are applicable to the unstratified case, and Lindzen *et al.* (1980) extended them to the baroclinic-instability problem.

4. Necessary conditions for over-reflection

Although this gives us an insight into the process behind the instability, it raises two more questions: under what circumstances does over-reflection occur at a critical level, and what is the physical mechanism responsible for it?

In answer to the first question, we already know that a necessary condition for over-reflection in the stratified problem is $Ri(z_c) < \frac{1}{4}$. As we noted at the beginning of §3, the solutions of (25) will usually be exponential in z near the critical level in this case. Based on observations in other cases too, Lindzen & Tung (1978) suggested that the following 'geometry' might be necessary for over-reflection of a wave approaching a critical level from below:

(i) there must be a region below the critical level in which the solutions of (2.5) are oscillatory in z ('wave region I');

(ii) the critical level must be separated from wave region I by a region in which the solutions are exponential in z (a 'trapping region');

(iii) there must be a second region, above the critical level, in which the solutions are oscillatory ('wave region II');

(iv) there must be a reflecting surface at the top of the wave region II.

The need for 'wave region I' is clear, since there can be no over-reflection if there are no waves to be reflected. We shall here show by example that neither the 'trapping region' of condition (ii) nor the reflecting surface of condition (iv) is strictly necessary, though each may play an important role in some cases. The 'wave region II' of condition (iii) does appear to be necessary, and we shall discuss it further in §5.

Consider first the following example, which is due to K. K. Tung (private communication). Suppose we have a linear shear

$$U(z) = \alpha z \tag{4.1}$$

and a parabolic stability profile

$$N^2(z) = \beta^2 z^2. \tag{4.2}$$

Then, for a stationary wave ($c = 0$)

$$Q(z) = -k^2 + \beta^2/\alpha^2 = \lambda^2, \tag{4.3}$$

which we assume is positive. Since Q is positive and constant everywhere, solutions are oscillatory everywhere, and the 'trapping region' of condition (ii) is absent (even though $Ri < \frac{1}{4}$ at the critical level). Taking the radiation condition (2.7 *c*) as the upper boundary condition, the reflecting surface of condition (iv) is also absent, but a solution of (2.5) is

$$\Psi(z) = \mathrm{e}^{\mathrm{i}\lambda z} \quad (\lambda > 0). \tag{4.4}$$

Since $U - c$ changes sign at $z = 0$, this represents an upward-moving wave in $z > 0$, a downward-moving one in $z < 0$. That is, we have waves leaving the critical level on both sides, without any incoming waves being present: the reflection coefficient is infinite.

The example may seem rather special, since there is no singularity in Q, even though there is a critical level.† However, other examples are easily found, and we give two here. The first is a slight variation on Tung's example. Again assume linear shear $U = \alpha z$, and let

$$N^2(z) = \begin{cases} \beta z^2 & (|z| \geqslant \delta), \\ \beta \delta^2 & (|z| \leqslant \delta), \end{cases} \tag{4.5}$$

so that for $c = 0$

$$Q(z) = \begin{cases} -k^2 + \dfrac{\beta^2}{\alpha^2} = \lambda^2 & (|z| > \delta), \\[2ex] -k^2 + \dfrac{\beta^2 \delta^2}{\alpha^2 z^2} = -k^2 + \dfrac{Ri}{z^2} & (|z| < \delta), \end{cases} \tag{4.6}$$

where $Ri = \beta^2 \delta^2 / \alpha^2$ is the Richardson number in $|z| < \delta$. We assume $\lambda^2 > 0$ and $Ri < \frac{1}{4}$. Then $|z| < \delta$ is a 'trapping region', so condition (ii) is satisfied, but with (2.7 *c*) as upper boundary condition, the reflecting surface required by condition (iv) is absent. Nonetheless, a solution of (2.5) is

$$\Psi(z) = \begin{cases} \mathrm{e}^{-\mathrm{i}\lambda z} + R\, \mathrm{e}^{\mathrm{i}\lambda z} & (z < -\delta), \\ A z^{\frac{1}{2}} I_\mu(kz) + B z^{\frac{1}{2}} I_{-\mu}(kz) & (|z| < \delta), \\ T\, \mathrm{e}^{\mathrm{i}\lambda z} & (z > \delta), \end{cases} \tag{4.7}$$

where $\mu = (\frac{1}{4} - Ri)^{\frac{1}{2}}$, $I_{\pm\mu}$ are modified Bessel functions (see e.g. Abramowitz & Stegun 1970), §9.6) and R and T are the (complex) reflection and transmission coefficients associated with the region $|z| < \delta$, i.e. with the critical level and its associated 'trapping region'. The constants A, B, R and T are determined by the requirements that Ψ and Ψ_z should be continuous at $z = \pm\delta$. After some algebra, one finds that

$$\left.\begin{aligned} R &= \frac{-[kI'_\mu - (\mathrm{i} - \frac{1}{2}\delta) I_\mu][kI'_{-\mu} - (\mathrm{i}\lambda - \frac{1}{2}\delta) I_{-\mu}]\, \mathrm{e}^{2\mathrm{i}\lambda\delta}}{k^2 I'_\mu I'_{-\mu} + (\lambda^2 + \frac{1}{4}\delta^2) I_\mu I_{-\mu} + \dfrac{2\lambda}{\pi\delta}\cos\mu\pi + \dfrac{k}{2\delta}(I_\mu I'_{-\mu} + I'_\mu I_{-\mu})}, \\ T &= \frac{2\mathrm{i}\lambda/\pi\delta}{k^2 I'_\mu I'_{-\mu} + (\lambda^2 + \frac{1}{4}\delta^2) I_\mu I_{-\mu} + \dfrac{2\lambda}{\pi\delta}\cos\mu\pi + \dfrac{k}{2\delta}(I_\mu I'_{-\mu} + I'_\mu I_{-\mu})}, \end{aligned}\right\} \tag{4.8}$$

† In anticipation of results later in this paper, it should be noted that the kinematic role of the shear at a critical level does not depend on any singularity.

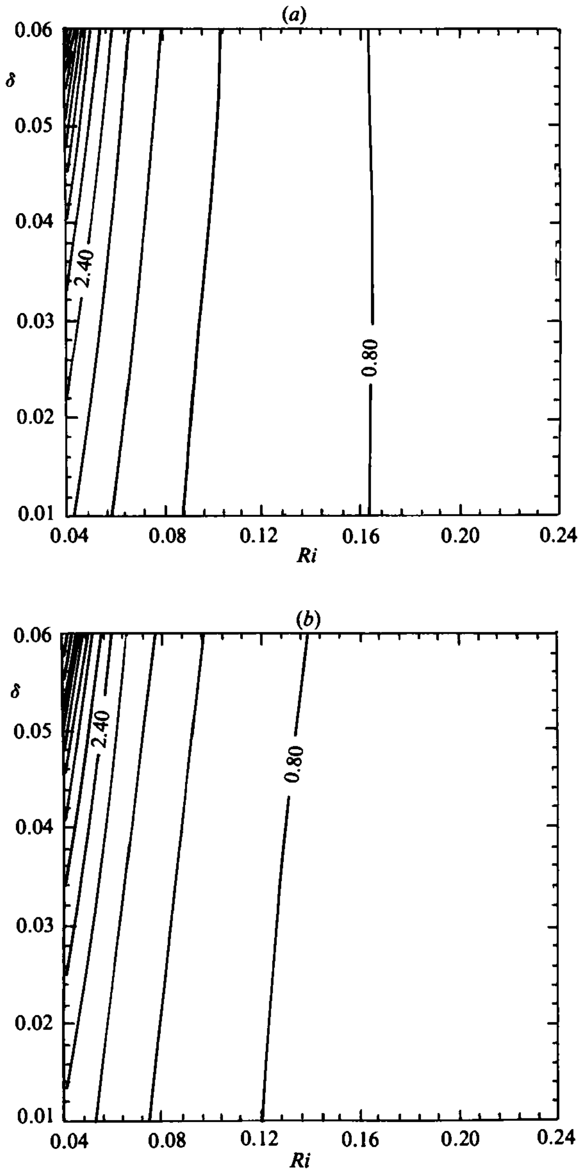


FIGURE 1. (*a*) Reflection coefficient as a function of Richardson number (Ri) at the critical level and width δ of the trapping region for the modified Tung example. From (4.8) with $k = 2.22$. (*b*) Transmission coefficient for the same example.

where the argument $k\delta$ is understood for the Bessel functions. $|R|$ and $|T|$ are plotted against Ri and δ in figure 1. Both exceed 1 for certain values of Ri and δ.

As a second example, again assume $U = \alpha z$, but take

$$N^2(z) = \begin{cases} N_1^2 & (|z| > \delta), \\ N_2^2 & (|z| < \delta), \end{cases} \tag{4.9}$$

$$Q(z) = \begin{cases} -k^2 + Ri_1/(z-\tilde{c})^2 & (|z| > \delta), \\ -k^2 + Ri_2/(z-\tilde{c})^2 & (|z| < \delta), \end{cases} \tag{4.10}$$

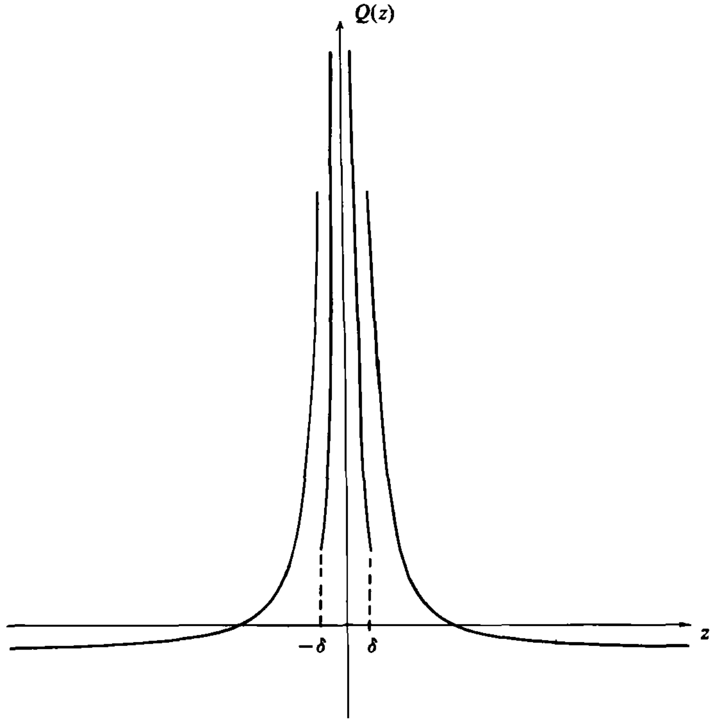


FIGURE 2. Index of refraction $Q(z)$ for the second example of §4. From (4.10), with $\tilde{c} = 0$ and representative values of the other parameters.

where $\tilde{c} = c/\alpha$. $Ri_1 = N_1^2/\alpha^2$ and $Ri_2 = N_2^2/\alpha^2$. Q is plotted in figure 2. Assume $Ri_2 < \frac{1}{4}$, and $|\tilde{c}| < \delta$, so that the critical level lies within the 'trapping region' $|z| < \delta$. Assume also $Ri_1 > \frac{1}{4}$, so that solutions are oscillatory outside $|z| < \delta$, and the two wave regions are present. Since $Q < 0$ for $|z-\tilde{c}| > Ri_1^{\frac{1}{2}}/k$, the appropriate boundary conditions are (2.7*b*), $\Psi \to 0$ as $z \to \pm\infty$.

Unfortunately, the solution of this problem is not one we are interested in. The turning points at $|z-\tilde{c}| = Ri_1^{\frac{1}{2}}/k$ act as reflecting surfaces, and the solution will represent a standing wave in each of the two wave regions $-\tilde{c}-Ri_1^{\frac{1}{2}}/k < z < -\delta$ and $\delta < z < -\tilde{c}+Ri_1^{\frac{1}{2}}/k$. There will thus be waves incident on the critical level from both sides, and it will not be possible to deduce its reflection and transmission coefficients. What we are interested in is a solution representing a wave incident on the critical level from one side only, together with the resulting reflected and transmitted waves. Such a solution is

$$\Psi(z) = \begin{cases} e^{-\mu_1\pi}(z-\tilde{c})^{\frac{1}{2}} I_{i\mu_1}(k(z-\tilde{c})) + R\, e^{\mu_1\pi}(z-\tilde{c})^{\frac{1}{2}} I_{-i\mu_1}(k(z-\tilde{c})) \quad (z < -\delta), \\ A(z-\tilde{c})^{\frac{1}{2}} I_{\mu_2}(k(z-\tilde{c})) + B(z-\tilde{c})^{\frac{1}{2}} I_{-\mu_2}(k(z-\tilde{c})) \quad (|z| < \delta), \\ T(z-\tilde{c})^{\frac{1}{2}} I_{i\mu_1}(k(z-\tilde{c})) \quad (z > \delta), \end{cases} \tag{4.11}$$

where $\mu_1 = (Ri_1 - \frac{1}{4})^{\frac{1}{2}}$ and $\mu_2 = (\frac{1}{4} - Ri_2)^{\frac{1}{2}}$. This solution does not satisfy the boundary conditions, of course. One may just ignore this, or assume that Q is modified for $|z| > \delta+\epsilon$ $(\epsilon > 0)$ in such a way as to allow (4.11) to be the exact solution in $|z| < \delta+\epsilon$, or assume the fluid to be bounded at $z = \pm(\sigma+\epsilon)$ by perfectly absorbent walls, at the lower end of which an upward-moving wave of arbitrary amplitude may be generated.

In any event, it is this solution (4.11) that enables us to find the reflection and transmission coefficients R and T of the 'trapping region' $|z| < \delta$, which contains the critical level. Continuity of Ψ and Ψ_z at $z = \pm\delta$ leads to

$$\left.\begin{aligned} R &= \frac{-W[\mathrm{i}\mu_1,\mu_2;\zeta_-]\,W[\mathrm{i}\mu_1,-\mu_2,\zeta_+]\,\mathrm{e}^{\mathrm{i}\mu_2\pi} - W[\mathrm{i}\mu_1,-\mu_2;\zeta_-]\,W[\mathrm{i}\mu_1,\mu_2;\zeta_+]\,\mathrm{e}^{-\mathrm{i}\mu_2\pi}}{W[\mathrm{i}\mu_1,-\mu_2;\zeta_-]\,W[\mathrm{i}\mu_1,\mu_2,\zeta_+]\,\mathrm{e}^{-\mathrm{i}\mu_1\pi} - W[\mathrm{i}\mu_1,\mu_2;\zeta_-]\,W[-\mathrm{i}\mu_1,-\mu_2;\zeta_+]\,\mathrm{e}^{-\mathrm{i}\mu_2\pi}}, \\ T &= \frac{\zeta_+\{W[-\mathrm{i}\mu_1,\mu_2;\zeta_+]\,W[\mathrm{i}\mu_1,-\mu_2;\zeta_+] - W[-\mathrm{i}\mu_1,-\mu_2;\zeta_+]\,W[\mathrm{i}\mu_1,\mu_2;\zeta_+]\}}{\zeta_-[\{W[\mathrm{i}\mu_1,-\mu_2;\zeta_-]\,W[-\mathrm{i}\mu_1,\mu_2;\zeta_+]\,\mathrm{e}^{-\mathrm{i}\mu_2\pi} - W[\mathrm{i}\mu_1,\mu_2;\zeta_-]\,W[-\mathrm{i}\mu_1,-\mu_2;\zeta_+]\,\mathrm{e}^{\mathrm{i}\mu_2\pi}\}} \end{aligned}\right\} \qquad (4.12)$$

where $W[\mu,\nu;\zeta] = I_\mu(\zeta)\,I'_\nu(\zeta) - I'_\mu(\zeta)\,I_\nu(z)$, and $\zeta_\pm = k(\delta \pm \tilde{c})$. Various standard properties of the Bessel functions I_ν have been used to obtain (4.12); they may be found in Abramowitz & Stegun (1970, pp. 374–379) for example. $|R|$ and $|T|$ are plotted against Ri_1 and Ri_2 (at $\tilde{c} = 0$, $k = 1$, $\delta = 0.1$), and against complex $\tilde{c}$ (at $Ri_1 = 0.5$, $Ri_2 = 0.05$) in figure 3. Again, both exceed 1 for certain parameter values.

As we noted, both these examples do possess the 'trapping region' required by condition (ii), even though the example of Tung shows that it is not strictly necessary for over-reflection. Except in the special cases where $N^2 = 0$ or $U_z = 0$ at the critical level, however, the condition $Ri(z_c < \frac{1}{4})$, which is necessary, guarantees that the critical level will be embedded in a 'trapping region'. Its role may be to ensure that the disturbance reaches the critical level, for, as shown by Bretherton (1966*b*), the group velocity of a wave falls to zero as the critical level is approached, and so if the wave can propagate right up to the critical level it may take infinitely long to do so. Only in the special cases, which include Tung's example, does the group velocity remain non-zero at the critical level. Presumably, this is what eliminates the need for an exponential 'tunnelling' region.

All these examples given show that the reflecting surface of condition (iv) is not necessary for over-reflection. This has already been noted by Tai (1983). However, such a surface may play an important role in producing over-reflection and consequent instability. For suppose that, in the example (4.6), a solid wall is placed at some point z_T above the critical level ($z_T > \delta$). Neglecting transient effects, consider what happens when a wave of unit amplitude is incident from below the critical level. A reflected wave will arise, with amplitude R given by (4.8). Also, an upward-moving wave of amplitude T, given by (4.8), will appear in the wave region above the critical level. This transmitted wave will be reflected off the wall at z_T, back down to the critical level. Assume for convenience that $c = 0$, so that Q is symmetric in z and the reflection and transmission coefficients for a wave incident from above the critical level will be the same as those for one incident from below. The transmitted wave, of amplitude T, will then give rise to a downward-moving wave of amplitude T^2 below the critical level, adding to the directly reflected wave of amplitude R already there, and an upward-moving wave of amplitude TR above the critical level, adding to the directly transmitted wave of amplitude T already there. This second upward-moving wave will again bounce off the wall at z_T, return to the critical level, and produce another downward-moving wave below the critical level, of amplitude RT^2, and another upward-moving wave above the critical level, of amplitude TR^2. This bouncing about in the upper wave region will continue indefinitely, each bounce producing another contribution to the total downward-moving wave below the critical level, i.e. another contribution to the 'net' reflection coefficient. The final

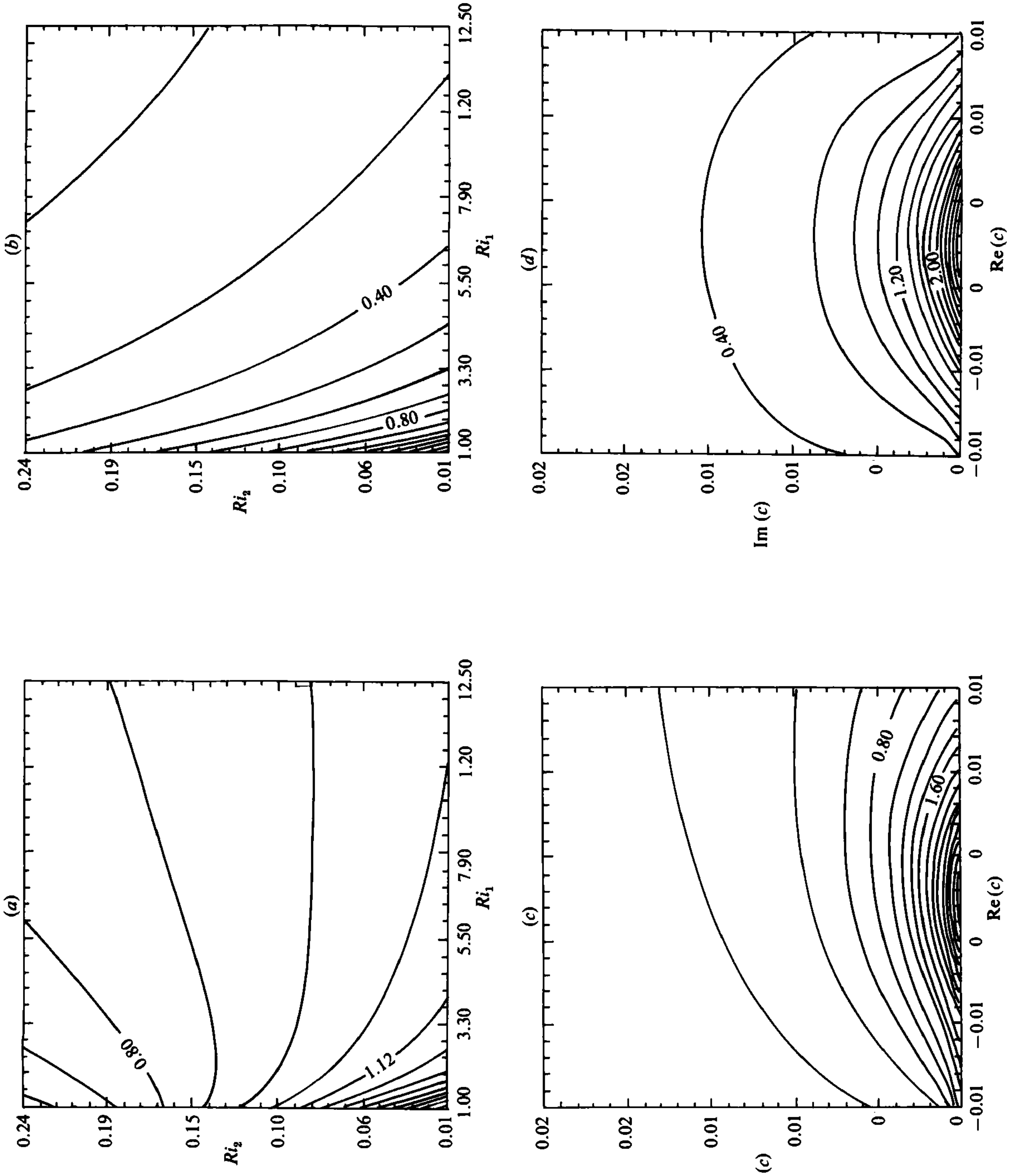


FIGURE 3. For description see facing page.

value of this 'net' reflection coefficient is thus

$$R' = R + T^2 e^{i\theta}(1 + R e^{i\theta} + R^2 e^{2i\theta} + \dots), \tag{4.13}$$

where θ is the change in phase experienced by a wave propagating across the upper wave region and back (including reflection at z_T). If $|R| < 1$

$$R' = R + \frac{T^2 e^{i\theta}}{1 - R e^{i\theta}}. \tag{4.14}$$

Even if both $|R|$ and $|T|$ are less than one, $|R'|$ may be greater than one. In fact, if $|R| + |T| > 1$, it may be possible to choose θ (i.e. z_T) so that $|R'| > 1$; whether or not this is the case depends on $\arg(R^{-1}T)$. This has a clear interpretation in terms of energy; if $|R| + |T| > 1$ the critical level is a source of wave energy, and one only needs to contain this energy, avoiding destructive interference, in order to get over-reflection.

If $|R| \geqslant 1$ and $T \neq 0$ the series in (4.13) diverges, so $|R'|$ is infinite. In this case the presence of the upper boundary leads to instability in the manner proposed by Lindzen and described in §3. It should be noted, however, that the boundary-value problem (2.5), (4.6), (2.7*a*) always has a solution with $c = 0$, and in fact this solution will have a reflection coefficient given approximately† by (4.14) regardless of the magnitude of $|R|$. If $|R| > 1$ therefore, the solution of the boundary-value problem is misleading; it will give a solution that is unstable and will not be observed in practice. Fortunately, the procedure used in solving the boundary-value problem makes it easy to avoid overlooking such an instability. For suppose the instability has $c = c_0$ ($\mathrm{Im}(c_0) > 0$). Then $R = \infty$ at $c = c_0$, and it will usually be the case that $|R|$ increases as $\mathrm{Im}(c)$ increases from zero to $\mathrm{Im}(c_0)$. This is the reverse of the situation when there is no instability, when the fact that $\mathrm{Im}(c)$ represents a damping will cause $|R|$ to decrease as $\mathrm{Im}(c)$ increases. Since, in the numerical procedure, one must solve the problem at a sequence of values of $\mathrm{Im}(c)$ (see Appendix A), it will be easy to see whether $|R|$ increases or decreases with increasing $\mathrm{Im}(c)$, and so whether or not there is an instability.

All the examples so far given satisfy condition (iii): they have a 'wave region II'. Indeed, extensive numerical searching has failed to produce any example of over-reflection without such a region (in an inviscid problem). Also, Rosenthal (1981) has shown that if

$$U(z) = \alpha z, \quad N^2(z) = \beta^2, \quad \zeta \geqslant -\delta \tag{4.15}$$

and $Ri = \beta^2/\alpha^2 < \frac{1}{4}$, so that the entire region $z \geqslant -\delta$ is 'trapping', then the reflection coefficient R for a mode with $c = 0$ always has $|R| \leqslant 1$. This is true either if there is a rigid lid at some height above the critical level, or if the fluid is unbounded above. It is not hard to extend this result to the case

$$U = \alpha z, \quad N^2(z) = \begin{cases} \beta_1^2 & (\delta \leqslant z \leqslant \delta_1), \\ \beta_2^2 & (\delta_1 < z), \end{cases} \tag{4.16}$$

where $\beta_1^2/\alpha^2 < \frac{1}{4}$ and $\beta_2^2/\alpha^2 < \frac{1}{4}$, which again is 'trapping' for all $z > -\delta$.

The evidence, then, is strongly suggestive of the necessity of a second wave region. We shall discuss the possible role of this region in §5.

† Since transient effects have been neglected in deriving (4.14).

FIGURE 3. (*a*) Reflection coefficient as a function of Richardson number at the critical level (Ri_2) and outside the trapping region (Ri_1) for the second example of §4. From (4.12) with $\tilde{c} = 0$, $k = 1$, $\delta = 0.01$. (*b*) Transmission coefficient for the same example. (*c*) Reflection coefficient as a function of complex c for the same example, with $Ri_1 = 0.5$ and $Ri_2 = 0.05$. (*d*) Transmission coefficient for the same example.

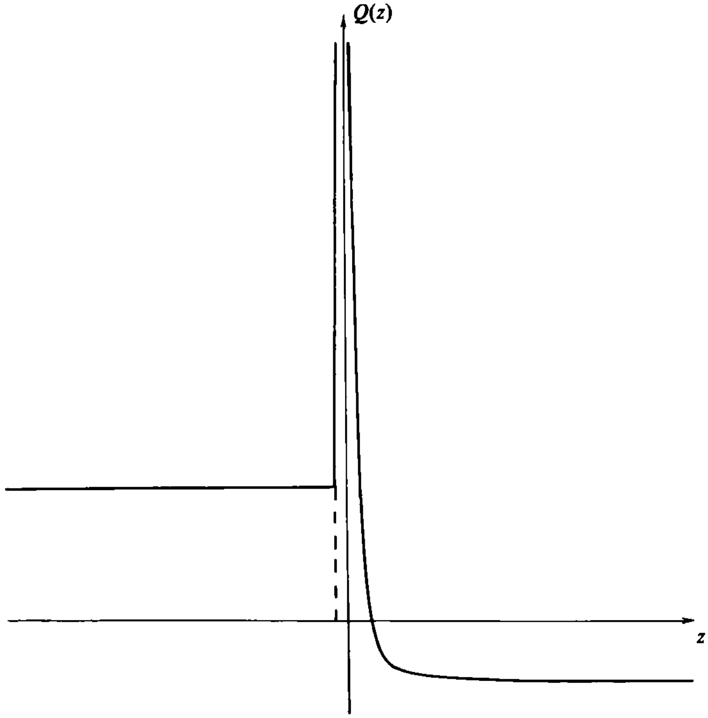


FIGURE 4. Index of refraction $Q(z)$, lacking a wave region above the critical level. From (5.1), with representative parameter values.

5. The role of the second wave region

It is known, from consideration of the wave fluxes, that, if there is a region $z \geqslant z_0$, extending to infinity or to a rigid upper boundary, in which solutions of (2.5) are exponential in character, and which contains no critical levels, then there will be perfect reflection ($|R| = 1$) from the bottom of this region. The solution below z_0 is thus independent of the details of the basic flow above z_0. We speculate that, in the case of a critical level above which there is no second wave region, the same may be true; a wave incident from below will be reflected with little regard for the details of the flow above the turning point z_0. It will thus not 'see' the critical level, and will not be over-reflected.

The role of the second wave region would then be to force some wave flux to pass through the critical level, allowing the wave to interact properly with, or 'see', it. The fact that there is no need for wave containment in this region is consistent with this suggestion. If our suggestion were correct, anything which causes a wave flux through the critical level should be able to produce over-reflection. In particular, a region of linear damping above the critical level (in this otherwise inviscid problem), which would be a sink of wave flux, would draw a flux through the critical level from below, and should therefore be able to cause over-reflection just as a second wave region does.

To test this we take

$$U(z) = \alpha z, \quad N^2(z) = \begin{cases} \beta^2 & (z \geqslant -\delta), \\ \beta^2 (z/\delta)^2 & (z \leqslant -\delta), \end{cases} \tag{5.1}$$

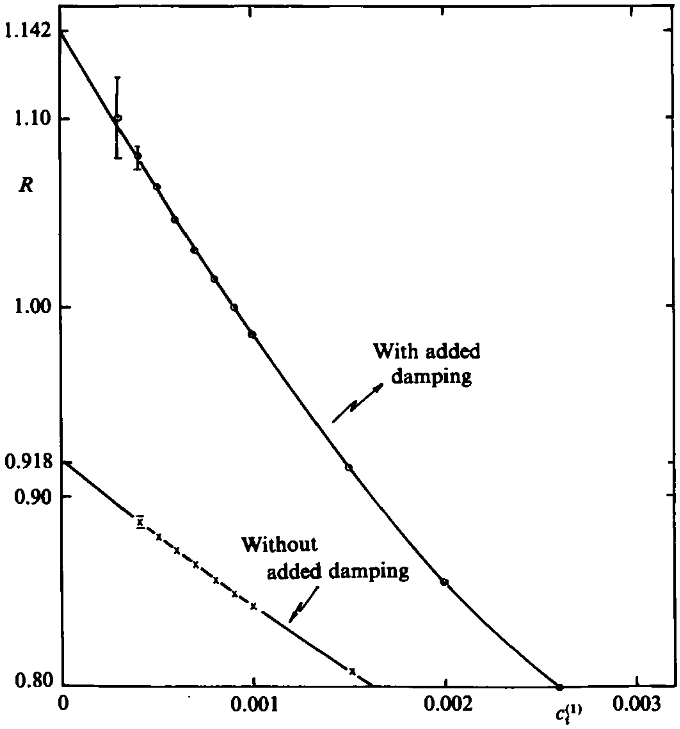


FIGURE 5. Reflection coefficient computed numerically for the profile of (5.1) with no added damping (lower curve) and with damping given by (5.2) (upper curve), plotted against the imaginary part of c introduced for computational reasons. The parameters used were $c_r = 0$, $k = 2.22$, $\delta = 0.26$, $\alpha = 1$, $\beta^2 = 0.01$, $\mu = 0.028$, $\sigma = 0.015$ and $\epsilon = 0.035$.

where $Ri = \beta^2/\alpha^2 < \frac{1}{4}$. The resulting Q is shown for $c = 0$ in figure 4. There is no wave region above the critical level. The solution of (2.5) with (2.7*b*) as upper boundary condition is found numerically, as described in Appendix A. Because of the singularity in Q at $U = c$, one must let c have a non-zero imaginary part ic_i when performing the numerical calculation. The derived result is obtained by repeating the calculation for a series of successively smaller values of c_i and extrapolating to $c_i = 0$. Care must be taken to ensure that sufficiently small values of c_i are taken, and that sufficiently many grid points are taken to resolve the solution adequately.

The damping is also included in the imaginary part of c, so that when all terms are included

$$c = c_r + ic_i, \quad c_i = c_i^{(1)} + \epsilon g(z), \tag{5.2}$$

where $c_i^{(1)}$ is the constant inserted for numerical reasons, ϵ is a measure of the amplitude of the damping, and $g(z)$ determines its distribution. We use

$$g(z) = \exp\left\{-\frac{(z-\mu)^2}{2\sigma}\right\}. \tag{5.3}$$

In figure 5 we show the computed value of the reflection coefficient R against $c_i^{(1)}$, both without damping ($\epsilon = 0$) and with damping ($\epsilon = 0.035$). The other parameters used were $\mu = 0.028$, $\sigma = 0.015$, $c_r = 0$, $\delta = 0.026$, $\alpha = 1$, $\beta^2 = 0.01$ and $k = 2.22$. It can be seen that the addition of damping raises the reflection coefficient from approximately 0.92 to approximately 1.15. The important thing is that it is raised above 1.

Thus a region of damping can play the role of the second wave region in producing over-reflection, and this supports our suggestion that its role is simply to draw some wave flux through the critical level.

This consequence of introducing damping may be of interest in another context, as fluid flows with viscosity are almost always unstable when the viscosity is sufficiently small, even when the inviscid analysis indicates stability. A resolution for this apparent inconsistency was suggested by Taylor (1915), who noted that a finite amount of momentum could be transferred to the fluid from its boundaries in the presence of even an infinitesimal amount of viscosity. Our new result suggests that the viscous stability problem also may be understood in terms of over-reflection. For example, in (unstratified) Poiseuille flow

$$U(z) = z(1-z) \tag{5.4}$$

a neutral mode with $0 < c < \frac{1}{4}$ has two critical levels, between which there is a wave region. There is, however, no over-reflection, because $Q < 0$ between the critical levels and the walls at $z = 0$ and 1, so there is no 'wave region II' for either critical level. Since $U_{zz} = -\frac{1}{2}$ is one-signed, the flow is stable in the inviscid case (Rayleigh's theorem), but it is known to be unstable in the viscous case for any sufficiently small viscosity. Since the viscosity is of the form $-\nu\Psi_{zz}$, it acts as a concentrated damping near the boundaries. Also, by permitting diffusion across the critical level, the viscosity simulates a 'trapping region' between the wave region and the critical level which would otherwise be absent. Lindzen and Rambaldi recently completed calculations which confirm the existence of over-reflection in this problem.

6. Time development of over-reflection

The time development of over-reflection has not been studied very much in the past. It is of interest for, as noted by Lindzen & Rosenthal (1976), it will certainly affect the growth rate of an instability arising from over-reflection. Moreover, it was our hope that watching the evolution of over-reflection would be helpful in discovering how the mechanism operates. McIntyre & Weissman (1978) considered a case of resonant (infinite) over-reflection in a Helmholtz velocity profile, and found that the reflected and transmitted waves grew as the integral of the incident wave. For a constant-amplitude incident wave this would mean linear growth. The discontinuous Helmholtz velocity profile is a special case, however, and it is not apparent whether their result has any generality. Moreover, much of what happens occurs, for the Helmholtz profile, in corners where details are obscured.

In this section we present results of a few experimental integrations of an initial-value problem. The equations of motion are (2.3), but we assume solutions of the form

$$(\psi(x,z,t),\ \omega(x,z,t),\ \rho(x,z,t)) = \mathrm{e}^{\mathrm{i}kx}(\Psi(z,t),\ \Omega(z,t),\ R(z,t)), \tag{6.1}$$

so that the equations become

$$\left(\frac{\partial}{\partial t}+\mathrm{i}kU\right)\Omega+\mathrm{i}kU_{zz}\Psi = -\mathrm{i}kR, \quad \left(\frac{\partial}{\partial t}+\mathrm{i}kU\right)R+\mathrm{i}kN^2\Psi = 0, \quad \left(\frac{\partial^2}{\partial z^2}-k^2\right)\Psi = -\Omega. \tag{6.2}$$

We solve these equations numerically using a scheme due to Hyman (1979), described in Appendix B.

For the basic state we would like to use the first of the examples given in §4, with linear shear and parabolic stability (4.5). Unfortunately, since N^2 increases rapidly with $|z|$, the vertical group velocity of a wave becomes large at large $|z|$, and this would require us to use a prohibitively short time step in the numerical calculation. To avoid this we set N^2 = constant for $|z|$ greater than a particular value z_2, and modify U so that it too is constant for large $|z|$, ensuring that Q is a positive constant for $z \geqslant z_4$. We thus have uniform 'run-up' regions above and below the critical level. The actual choice we make is

$$U(z) = \begin{cases} \alpha z & (0 \leqslant z \leqslant z_3), \\ \tfrac{1}{2}\alpha(z_3+z_4)+\alpha(z_4-z_3)f\left(\dfrac{z-z_4}{z_3-z_4}\right) & (z_3 \leqslant z \leqslant z_4), \\ \tfrac{1}{2}\alpha(z_3+z_4) & (z \geqslant z_4), \end{cases} \tag{6.3a}$$

with $U(-z) = -U(z)$ and

$$f(x) = \begin{cases} \tfrac{2}{3}x^3 & (0 \leqslant x \leqslant \tfrac{1}{2}), \\ -\tfrac{2}{3}x^3+2x^2-x+\tfrac{1}{6} & (\tfrac{1}{2} \leqslant x \leqslant 1), \end{cases}$$

and

$$N^2(z) = \begin{cases} N_2^2 & (0 \leqslant |z| \leqslant z_1), \\ \dfrac{N_2^2}{N_1^2}z^2 & (z_1 \leqslant |z| \leqslant z_2), \\ N_1^2 & (|z| \geqslant z_2), \end{cases} \tag{6.3b}$$

where $z_2 = (N_1/N_2)z_1$. These profiles are shown, together with the resulting q, in figure 6.

The computational domain used, $-z_T \leqslant z \leqslant z_T$, is made as large as possible, because it is impossible to prevent partial reflection of an outgoing wave at the boundaries. The computation will be of interest only until reflections from the boundaries return to the region near the critical level. For an upper boundary condition we use an approximate radiation condition

$$\Psi_z + \mathrm{i}(Q(z_T))^{\frac{1}{2}}\Psi = 0 \quad \text{at } z = z_T, \tag{6.4a}$$

and we 'shake' the bottom boundary to generate an upward-moving wave:

$$\Psi(-z_T, t) = \exp(\mathrm{i}\mu t). \tag{6.4b}$$

This produces a wave with horizontal phase speed $c = -\mu/k$. One may take $\mu = 0$, or equivalently redefine $u = u+c$ or $z = z-c$ to ensure that the critical level lies at $z = 0$. At $t = 0$ we take $\Omega = R = 0$.

We select $k = 2.22$, $\alpha = 1$, $z_3 = 0.9930$, $z_4 = 1.8559$ and $N_1^2 = 15$. This ensures that $Q > 0$ for all z, so that internal waves are supported everywhere, except perhaps in the neighbourhood $|z| < z_1$ of the critical level. Thus the two 'wave regions', one above and one below the critical level, will be present. By choosing $N_2^2 < \frac{1}{4}$, the Richardson number at the critical level $Ri_2 = N_2^2/\alpha^2$ will be $< \frac{1}{4}$, and the region $|z| < z_1$ will be a 'trapping' region.

The reflection and transmission coefficients are measured at the points $\pm z_R$, which are chosen to lie at the edges of the uniform run-up regions, i.e. z_R is slightly greater than z_4. Thus we are not in fact measuring them for the critical level alone, but for the whole region $-z_4 \leqslant z \leqslant z_4$, which includes the transition regions $|z| = z_2$ and $z_3 \leqslant |z| \leqslant z_4$ that will give rise to partial reflections. We find that the errors so introduced are not significant. As discussed in Appendix C, there are other difficulties

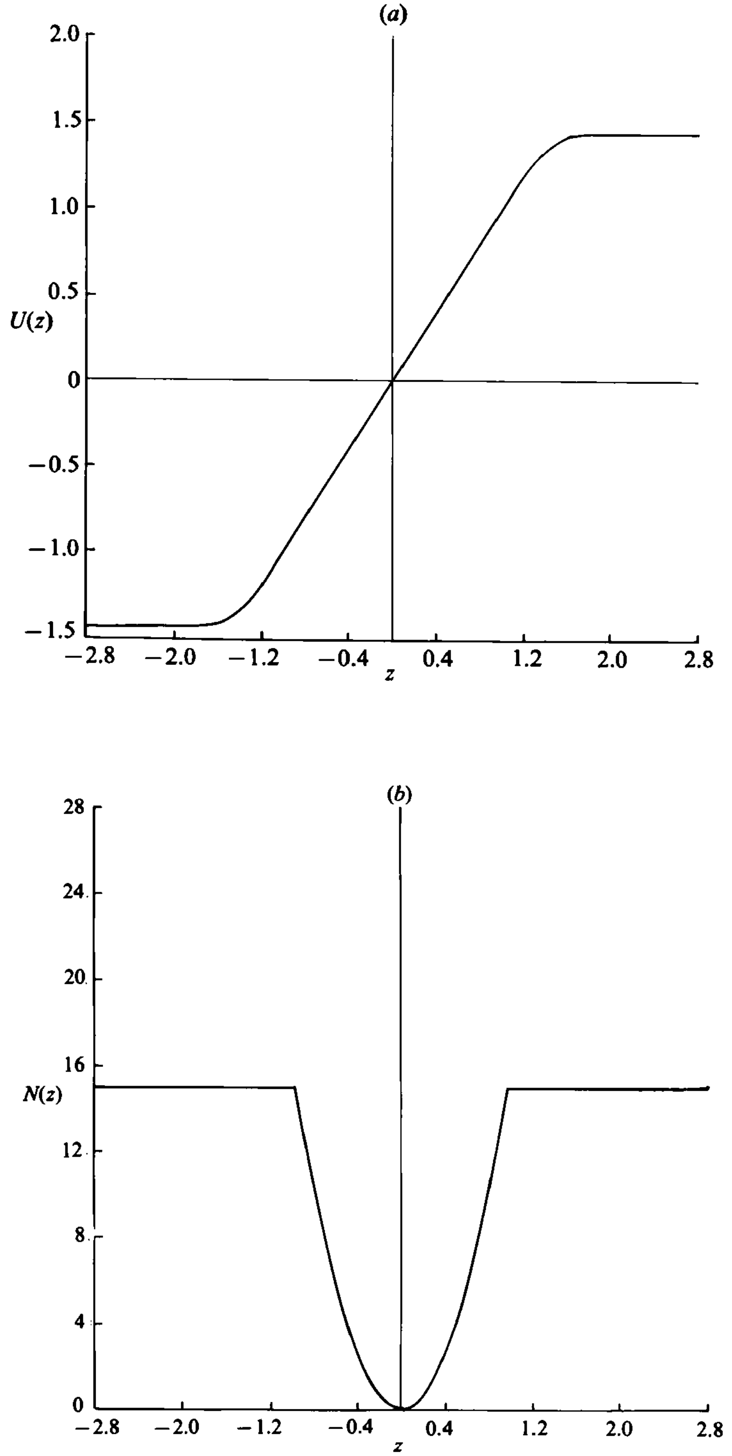


FIGURE 6 (*a*, *b*). For description see facing page.

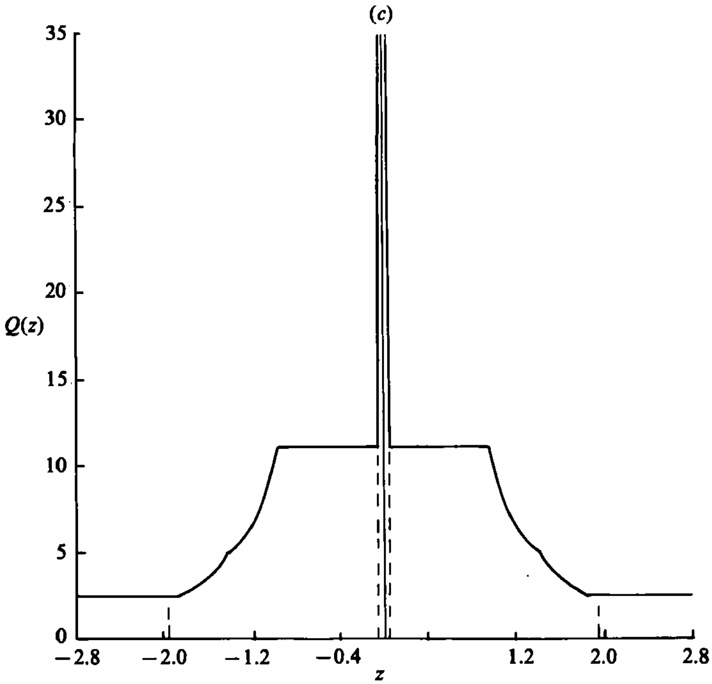


FIGURE 6. (*a*) Basic shear velocity $U(z)$ used in initial-value problem, from (6.3). (*b*) Static stability $N^2(z)$ for the same example. (*c*) Index of refraction $Q(z)$ for the same example.

associated with transient effects, but we find it adequate to use the obvious formulae (C 7) and (C 8), which reduce, when the incoming wave is assumed to have unit amplitude, to

$$R(t) = \frac{[\Psi_z - \mathrm{i}m\Psi]_{z=-z_\mathrm{R}}}{2m}, \quad T(t) = \frac{[\Psi_z - \mathrm{i}m\Psi]_{z=-z_\mathrm{R}}}{2m}, \tag{6.5}$$

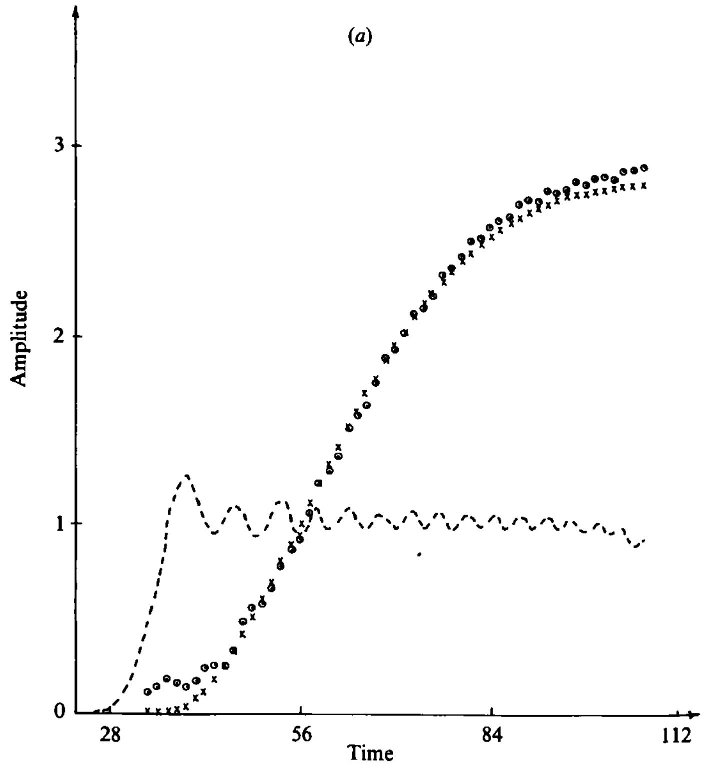
where $m = +(Q(z_\mathrm{R}))^{\frac{1}{2}}$.

The behaviour at the critical level should not depend much on the structure of Q away from a neighbourhood of $|z| \leqslant z_1$. We were able to verify this, at least for the range of parameters we considered. Thus R and T should depend principally on Ri_2, z_1 and c, as well as t.

As $t \to \infty$ we would expect the values of R and T to approach their values in the corresponding steady-scattering problem. Taking $c = 0$, they should be approximately those given by (4.8), plotted in figures 1 and 2, though there will be slight discrepancies as a result of partial reflections off the transition regions and the presence of a small amount of dissipation in the numerical scheme.

In figure 7, R and T are plotted against time for three runs, each made with different values for z_1 and Ri_2. Also shown is the amplitude of the incoming wave, i.e. the amplitude of the upgoing wave immediately below $z = -z_4$.

It can be seen from these figures that the amplitudes of the reflected and transmitted waves grow smoothly to their final values, which are approximately the same as the values obtained from the steady problem. The incoming wave displays a certain roughness, but the amplitude of the transmitted wave varies smoothly with time. The reflected wave displays some roughness, but this may be attributed to



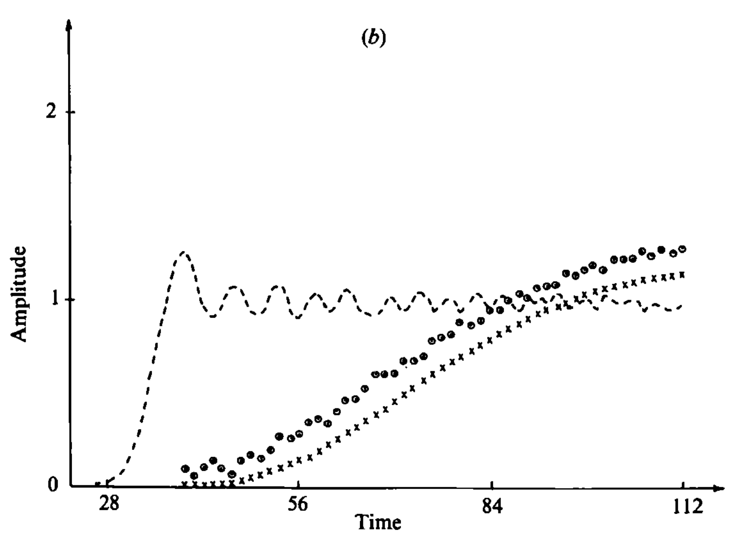


FIGURE 7 (*a*, *b*). For description see facing page.

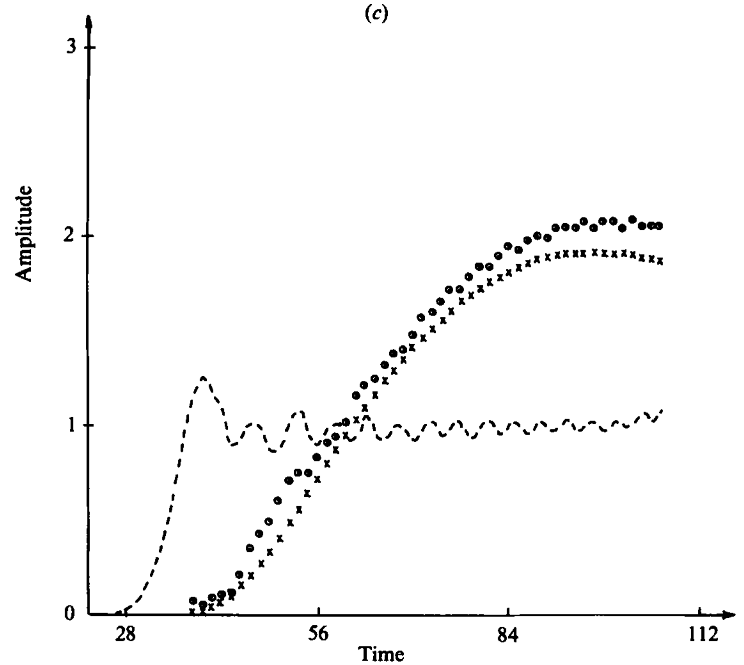


FIGURE 7. Results of the initial-value problem: incoming wave (dashed line), reflected wave (circles) and transmitted wave (crosses) for various cases. (*a*) $Ri_2 = 0.04$, $z_1 = 0.05$; (*b*) 0.08, 0.025; (*c*) 0.04, 0.0375.

partial reflections off the transition regions. Over much of the period of growth, the growth is approximately linear.

Both the rate of growth (of R and T) and the time taken to reach a steady value clearly depend on the parameters of the basic state. We find, for the nine runs we performed, that the growth rate is roughly proportional to z_1 at fixed Ri_2, and the constant varies roughly as $Ri_2^{-\frac{1}{2}}$. In dimensional form, then,

$$\frac{\mathrm{d}C}{\mathrm{d}t} = 0.07 \frac{k\,\Delta U}{Ri_2^{\frac{1}{2}}}, \tag{6.6}$$

where C is either R or T, and ΔU is the change in the basic velocity of the shear flow across the trapping region. All our results fit this formula to within 15%. Note that this result is of very similar form to the semicircle theorem, as might have been expected.

The time taken to reach a steady value shows no systematic variation with Ri_2, and all our results are fitted by

$$q = \frac{7 \pm 2}{k\,\Delta U}. \tag{6.7}$$

This is suggestive of the existence of a characteristic timescale for the development, independent of Ri_2. This in turn suggests that a kinematic process, involving only the action of shear on a wave disturbance, is responsible for the critical-level interaction. We shall discuss this further in §7.

As we noted in §4, in cases where $|R| < 1$ but $|R|+|T| > 1$ in the normal-mode problem, the inclusion of a rigid upper boundary may lead, via bouncing of the wave in the upper wave region, to a net reflection coefficient that exceeds one in magnitude.

In this case, the time taken for a wave to cross the upper wave region and return enters the problem as a *second* timescale, distinct from and in addition to the timescale associated with the critical level that is measured in this experiment.

Finally, we give an example to show that consideration of the time development of the reflection coefficient allows the growth rate of instabilities resulting from over-reflection to be predicted with greater accuracy. We take the first of the profiles we used in the initial-value problem, that is with $Ri_2 = 0.04$ and $z_1 = 0.05$. From the normal-mode problem we find that the reflection coefficient at the critical level for a wave with $c = 0$ is

$$R_0 = 3.05. \tag{6.8}$$

Next, we insert an upper boundary at $z = z_T$, requiring $\Psi(z_T) = 0$. Again we solve the normal-mode problem, looking for that value of c at which the reflection coefficient becomes infinite. We then vary z_T until the real part of this value is zero. Thus we find that when

$$z_T = 4.21 \tag{6.9}$$

the normal-mode problem has an unstable solution with

$$c_r = 0, \quad c_i = 0.013. \tag{6.10}$$

This is the true growth rate of the instability.

From the wave-reflection viewpoint, the instability results from a wave bouncing back and forth between the upper boundary (where it is perfectly reflected) and the critical level (where it is over-reflected). We have selected z_T so that the bouncing waves will all be in phase. Neglecting transient effects, a crude estimate of the growth rate c_i is thus given by

$$R_0 = e^{2kc_i\tau}, \tag{6.11}$$

where τ is the time taken for the wave to propagate across the upper wave region. Assuming the speed of propagation is the local group velocity for a neutral wave, we find that $\tau = 8.42$, and so

$$c_i = 0.030, \tag{6.12}$$

which is more than double the true growth rate.

In order to take transient effects into account, we propose the following simple model. From the initial-value problem, we found $R(t)$ when the incoming wave was approximately a step function (figure 7*a*). Suppose the response is idealized as shown in figure 8(*a*), growing linearly from zero to its final value in a time q. Since the problem is linear, we deduce that the response to an incoming wave of amplitude $1/\epsilon$, which lasts only for $0 < t < \epsilon$, is

$$\rho_\epsilon(t) = \begin{cases} \dfrac{1}{\epsilon} R_0\left(\dfrac{t}{q}\right) & (0 \leqslant t \leqslant \epsilon), \\ \dfrac{R_0}{q} & (\epsilon \leqslant t \leqslant q), \\ \dfrac{1}{\epsilon} R_0\left(q - t + \dfrac{\epsilon}{q}\right) & (q \leqslant t \leqslant q+\epsilon), \end{cases} \tag{6.13}$$

which is shown in figure 8(*b*). Taking the limit as $\epsilon \to 0$, the (idealized) response to a δ-function input is found:

$$\rho(t) = \begin{cases} R_0/q & (0 \leqslant t \leqslant q), \\ 0 & (\text{other } t). \end{cases} \tag{6.14}$$

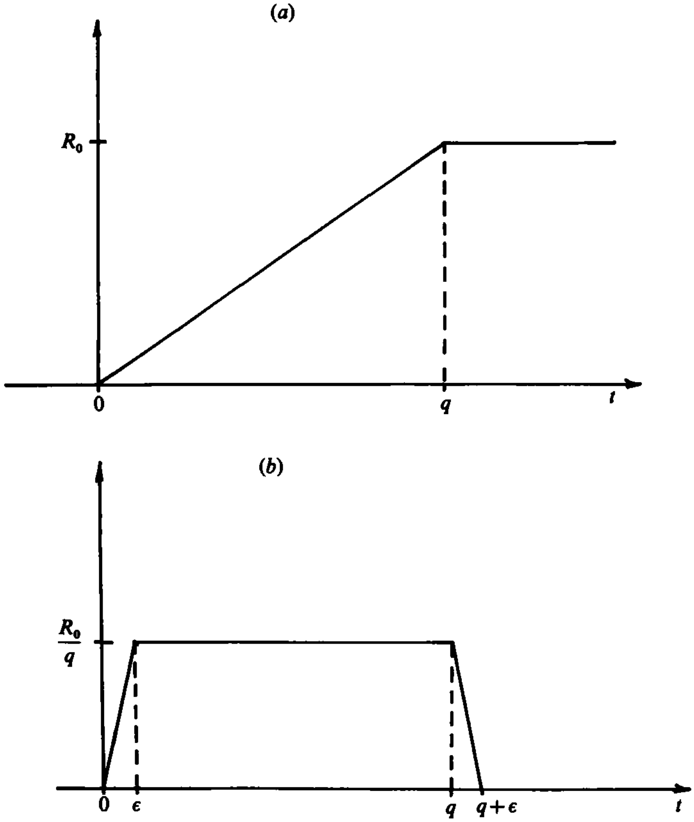


FIGURE 8. (*a*) Idealized time dependence of reflection coefficient of critical level when the incoming wave has unit amplitude for $t \geq 0$ (and zero amplitude for $t < 0$). (*b*) Amplitude of reflected wave when incoming wave has amplitude $1/\epsilon$ for $0 \leq t \leq \epsilon$ (and zero at other times).

Let $f(t)$ be the amplitude of the upgoing wave immediately above the critical level at time t. The amplitude of the downgoing wave there is then $f(t-2\tau)$. Thus

$$f(t) = \int_{-\infty}^{t} f(t'-2\tau)\,\rho(t-t')\,\mathrm{d}t' = \int_{t-q}^{t} f(t'-2\tau)\,\mathrm{d}t'\,R_0/q, \tag{6.15}$$

using (6.14)). Now let $f(t) = a\,\mathrm{e}^{kc_i t}$, then

$$a\,\mathrm{e}^{kc_i t} = \frac{aR_0}{q}\int_{t-q}^{t} \mathrm{e}^{kc_i(t'-2\tau)}\,\mathrm{d}t',$$

giving

$$\frac{R_0}{qkc_i}\,e^{-2kc_i\tau}(1-\mathrm{e}^{-kc_i q}) = 1. \tag{6.16}$$

The relation cannot be solved explicitly, but it is easily solved numerically. Using $q = 65$ (from figure 7*a*), we find a root of (6.14) at

$$c_i = 0.011, \tag{6.17}$$

which is clearly a much improved estimate of the true growth rate given by (6.10).

7. Concluding remarks

This study began, in fact, with the time-dependent calculations described in §6. Our hope was that a detailed description of the time evolution of wave over-reflection would help us understand over-reflection mechanistically. Initially we employed basic states wherein the upper wave region ('wave region II' of §4) was bounded – expecting the development of over-reflection to be associated with the wave-travel time in the upper wave region. Our finding that over-reflection developed in a *significantly shorter time than this*, led us to realize that wave containment in the upper wave region (condition (iv) of §4) was not essential. Thus most of our subsequent cases left wave region II open.

A review of our calculational results does suggest a mechanistic picture of wave over-reflection (and related instability). While some details remain to be proven, a brief description of the suggested picture may be of interest.

The picture consists in three parts. The first part relates to the interaction of a wave disturbance with the mean flow at a critical level.

This aspect of instability has been approached from a variety of seemingly disparate directions – primarily applied to the problem of unstratified shear flow. The first approach appears to be that of Orr (1907), who considered an initial perturbation of the form $\sin mz\, e^{ikx}$ (in the notation of the present paper) on a Couette flow ($\bar{u} = sz$). Despite the fact that unstratified Couette flow is stable with respect to normal modes, Orr found that his initial perturbation would initially grow before eventually decaying algebraically. Indeed the initial magnification became arbitrarily large as $m/k \rightarrow \infty$. Simultaneously the disturbance is advected by the mean flow; hence it, in effect, is at a critical level.

This appealing mechanism has been periodically reexamined (Goldstein 1938; Platzman 1952; Farrell 1982; Boyd 1983). The clearest physical explanation of the mechanism is given by Boyd. The crucial feature of the mechanism is the conservation of vorticity. Briefly, the initial perturbation must consist in, at least, a component in the form of a plane wave whose phaselines are oppositely tilted to the direction of the shear. Boyd then shows that the basic shear acts to rotate these phaselines in the direction of the shear. Up to the time when the phaselines are vertical (i.e. normal to the basic flow), the centres of maximum vorticity will become increasingly separated. But, since vorticity is conserved, and since vorticity consists in derivatives of velocity, velocity and kinetic energy must increase during this phase. For subsequent times, the centres are again brought arbitrarily close and the disturbance energy decays. No such simple picture has yet been produced for the stratified case; however, Brown & Stewartson (1980) show that similar mathematical behaviour obtains in this case, albeit with slightly different algebraic exponents. Presumably similar physics is operating. The general impression of the Orr mechanism has been that, since it leads to eventual disturbance decay, it is unrelated to normal mode instabilities. As we will soon show, this is not logically correct.

Bretherton's (1966*a*, *b*) approach to disturbance amplification at a critical level offer an important insight into this matter. Bretherton showed that a disturbance at a critical level will always extract energy from the mean flow. Bretherton's approach differs from Orr's in that he considers a continuously maintained disturbance; disturbance distortion and decay is not permitted. In essence, we see that by *maintaining* an appropriate disturbance at a critical level, *we can continuously excite the growing phase of the Orr mechanism.*

This maintenance of an appropriate disturbance at a critical level constitutes the second part of our picture. If there is any portion of the basic state that sustains wave propagation, one could imagine generating a wave incident on a critical level, which would, in turn, continuously excite the Orr mechanism, whose amplification would lead to over-reflection. As we have seen, the situation is not so simple. There are two aspects to the problem of a suitable wave disturbance actually being maintained at a critical level.

First (where relevant) we must get around the problem of group velocity going to zero at a critical level. This is done by interposing a region of exponential ('trapping') behaviour between the propagating region and the critical level – thus allowing the wave to 'tunnel' past the critical level. The existence of the exponential region is, by itself, insufficient. If there is no sink for the wave flux on the opposite side of the critical level then there will be no tunnelling – just exponential decay, and once again the wave disturbance will not reach the critical level. The second aspect consists therefore in providing a sink for the wave flux on the opposite side of the critical level. As shown in §5, this sink can be provided by either the second wave region or by a region of localized wave damping. Note that many of the well-known necessary conditions for instability are simply conditions for waves to reach the critical level. The Rayleigh inflection-point theorem and the Fjørtoft theorem together establish the complete necessary wave geometry (Lindzen & Tung 1978) while the Miles–Howard theorem ($Ri < \frac{1}{4}$) merely assures a trapping region wherein tunnelling can occur.

The above two parts provide our suggested scenario for the development of wave over-reflection and/or overtransmission. Note that in this case the disturbance at the critical level is being maintained by the continuously generated 'incident' wave. If, however, the wave region, where over-reflection is found, is bounded on the side of the wave region away from the critical level by a reflecting surface, then the over-reflected wave can be reflected back toward the critical level, leading to a *self-maintained* disturbance at the critical level. This leads us to the third part of our picture, which is specifically concerned with the development of unstable normal modes. The existence of the reflecting surface (which need not be perfectly reflecting – we only require that the product of reflection and over-reflection exceed unity) is not enough. In addition we need a 'quantization' condition: the thickness of the wave region must be such that the over-reflected and reflected waves have the appropriate phase relations to satisfy boundary conditions. These conditions have been discussed at length in Lindzen & Rosenthal (1976) and in Lindzen *et al.* (1980). The conditions are technical rather than basic, and reflect the fact that normal-mode instabilities are a special subset of the broader class of situations wherein waves entract energy from the mean flow. This third part, nevertheless, makes very clear the essential importance to shear instability of there being at least one wave region. The point is that waves carry a directed energy flux whose direction can be reversed at a reflecting surface. This in turn provides a convenient explanation of why viscous unstratified Couette flow is stable while viscous Poiseuille flow is unstable: Couette flow sustains wave propagation in no place, while Poiseuille flow sustains propagation in the middle of the parabolic flow.

The above scenario appears appropriate to all plane parallel flow instabilities: stratified and unstratified, viscous and inviscid. The specific application to viscous Poiseuille flow will be reported separately.

This work was supported by NSF Grant ATM-82-05638 and by NASA Grant NGL 22-007-228. Conversations with K. K. Tung, B. Farrell, A. J. Rosenthal and C. C. Lin are gratefully acknowledged.

Appendix A. Numerical solution of the normal-mode problem

To solve (2.5) numerically, we introduce a grid of points $z_\nu = z_B + (\nu - 1)\,\delta z$, $1 \leqslant \nu \leqslant N+1$, where $\delta z = (z_T - z_B)/(N-1)$, and z_B, z_T are the bottom and top of the computational domain, which includes the extra point z_{N+1} above the upper boundary. Then (2.5) is approximated by

$$\frac{\psi_{\nu+1} - 2\psi_\nu + \psi_{\nu-1}}{(\delta z)^2} + Q_\nu \psi_\nu = 0 \quad (2 \leqslant \nu \leqslant N), \tag{A 1}$$

where $Q_\nu = Q(z_\nu)$ and ψ_ν is an approximation to $\Psi(z_\nu)$. Introducing $\alpha_\nu = \psi_\nu / \psi_{\nu-1}$,

$$\alpha_\nu = [2 - (\delta z)^2 Q_\nu - \alpha_{\nu+1}]^{-1} \quad (2 \leqslant \nu \leqslant N). \tag{A 2}$$

Thus, once α_n or α_{N+1} is known, all the αs are easily found. If the upper boundary is a solid wall, then $\alpha_N = \psi_N = 0$. If the fluid extends to infinity, the upper boundary condition is either a radiation condition (if $Q_N > 0$) or a boundedness condition (if $Q_N < 0$), both of which may be written as

$$\Psi_z + (-Q(z))^{\frac{1}{2}}\,\Psi = 0 \quad \text{at } z = z_T, \tag{A 3}$$

which is approximated by

$$\psi_{N+1} - \Psi_{N-1} + 2\delta z(-Q_N)^{\frac{1}{2}}\,\Psi_N = 0, \tag{A 4}$$

or equivalently

$$\alpha_{N+1} - \alpha_N^{-1} + 2\delta z(-Q_N)^{\frac{1}{2}} = 0. \tag{A 5}$$

Eliminating α_{N+1} between (A 2) and (A 5) gives

$$\alpha_N = [1 + \delta z(-Q_N)^{\frac{1}{2}} - \tfrac{1}{2}(\delta z)^2 Q_N]^{-1}. \tag{A 6}$$

Once the α_ν are found, the ψ_ν may be found from

$$\psi_\nu = \alpha_\nu \psi_{\nu-1}, \tag{A 7}$$

where ψ_1 may be chosen arbitrarily to fix the amplitude of the solution.

Since Q becomes infinite where $U(z) = c$ for a neutral mode (with real c), one must introduce some artificial damping by letting c have a small positive imaginary part c_i. The desired solution is obtained by solving for a series of successively smaller values of c_i. Consequently a large number of grid points are needed to adequately resolve the solution near a critical level. Nonetheless, the amount of computing involved is small.

This method was used by, and described in, Lindzen *et al.* (1980).

Appendix B. Numerical solution of the initial-value problem

To solve the system (6.2), again introduce a grid z_ν and let $\Omega_\nu^{(n)}$, $R_\nu^{(n)}$ and $\Psi_\nu^{(n)}$ be the approximate solutions at grid point ν at time $t_n = n\,\delta t$. The solution can be advanced one time step in two stages. Given $\Omega_\nu^{(n)}$ and $R_\nu^{(n)}$, $1 \leqslant \nu \leqslant N$, one solves for $\Psi_\nu^{(n)}$ from an approximation to

$$\left.\begin{aligned} \Psi_{zz} - k^2\Psi &= -\Omega, \\ \psi(z_B) = \mathrm{e}^{\mathrm{i}\mu t}, \quad \Psi &\to 0 \quad \text{as } z \to \infty. \end{aligned}\right\} \tag{B 1}$$

We use the fourth-order-accurate Numerov scheme:

$$\left.\begin{aligned} D_+ D_-(\Psi_\nu^{(n)} - \tfrac{1}{12}k^2(\delta z)^2\,\Psi_\nu^{(n)}) - k^2\Psi_\nu^{(n)} &= -\Omega_\nu^{(n)} - \tfrac{1}{12}(\delta z)^2\,D_+ D_-\,\Omega_\nu^{(n)},\\ \Psi_\nu^{(n)} &= e^{i\mu t_n},\\ D_0\,\Psi_N^{(n)} + \tfrac{1}{6}k(\Psi_{N+1}^{(n)} + 4\Psi_N^{(n)} + \Psi_{N-1}^{(n)}) &= 0, \end{aligned}\right\} \tag{B 2}$$

from which $\Psi_{N+1}^{(n)}$ and $\Psi_1^{(n)}$ can be eliminated to leave a tridiagonal system.

Now that $\Psi^{(n)}$ is known, $R^{(n+1)}$ and $\Omega^{(n+1)}$ may be found from an approximation to

$$Y_t = \begin{pmatrix}\Omega_t\\ R_t\end{pmatrix} = -ik\begin{pmatrix}U\Omega + U_{zz}\,\Psi + R\\ UR + N^2\psi\end{pmatrix} \equiv G(Y). \tag{B 3}$$

Note these are ordinary differential equations in t, and no boundary conditions are needed (we take $\Omega = R = 0$ at $t = 0$). We use a predictor–corrector scheme due to Hyman (1979):

$$\left.\begin{aligned} \tilde{Y}_\nu^{(n+1)} &= Y_\nu^{(n-1)} + 2\,\delta t\,G_\nu(Y^{(n)}),\\ Y_\nu^{(n+1)} &= \tfrac{1}{5}\{4\,Y_\nu^{(n)} + Y_\nu^{(n-1)} + 4\,\delta t\,G_\nu(Y^{(n)}) + 2\,\delta t\,G_\nu(\tilde{Y}^{(n+1)})\}.\end{aligned}\right\} \tag{B 4}$$

Note that to evaluate $G_\nu(\tilde{Y}^{(n+1)})$ requires the $\tilde{\Psi}^{(n+1)}$ corresponding to the first approximation to $\tilde{\Omega}^{(n+1)}$ to be known. Thus (B 2) must be solved twice at each step. This is not a severe drawback. This scheme is chosen because of its excellent stability properties. It is third-order accurate. One may take a longer time step than allowed by a simple leapfrog scheme, and if desired may add dissipation without losing stability or the explicit nature of the scheme. The decoupling of odd and even time steps associated with leapfrog is also suppressed.

Appendix C. Reflection and transmission coefficients

In the normal-mode problem (2.5), the approximate WKB solution is

$$\Psi = AQ^{-\frac{1}{4}}\exp\left\{i\int^z Q^{\frac{1}{2}}\,dz\right\} + BQ^{-\frac{1}{4}}\exp\left\{-i\int^z Q^{\frac{1}{2}}\,dz\right\}, \tag{C 1}$$

where $Q^{\frac{1}{2}}$ has the same sign as $U-c$. From this, we may define the ratio of downward- to upward-moving waves at any point by

$$R = \left|\frac{B}{A}\right| = \left|\frac{\Psi_z - (-Q'/4Q + iQ^{\frac{1}{2}})\,\Psi}{\Psi_z - (-Q'/4Q + iQ^{\frac{1}{2}})\,\Psi}\right|, \tag{C 2}$$

which may be found from the numerical solution described in Appendix A:

$$\begin{aligned} R &\approx \left|\frac{\psi_{\nu+1} - \psi_{\nu-1} - 2\delta z(-Q_\nu'/4Q_\nu + iQ_\nu^{\frac{1}{2}})\,\psi_\nu}{\psi_{\nu+1} - \psi_{\nu-1} - 2\delta z(-Q_\nu'/4Q_\nu + iQ_\nu^{\frac{1}{2}})\,\psi_\nu}\right|\\ &= \left|\frac{(\alpha_{\nu+1} - \alpha_\nu^{-1}) - 2\delta z(-Q_\nu'/4Q_\nu + iQ_\nu^{\frac{1}{2}})\,\psi_\nu}{(\alpha_{\nu+1} - \alpha_\nu^{-1}) - 2\delta z(-Q_\nu'/4Q_\nu + iQ_\nu^{\frac{1}{2}})\,\psi_\nu}\right|. \end{aligned} \tag{C 3}$$

If there is no source of waves above z_ν, R is the reflection coefficient of the region $z \geqslant z_\nu$.

The ratio of upward-moving waves at two different points z_1 and z_2 is

$$T = \left|\frac{A_2}{A_1}\right| = \left|\frac{\Psi_z - (-Q/4Q - iQ^{\frac{1}{2}})\,\Psi}{Q^{\frac{1}{2}}}\right|_{z_2}\left|\frac{Q^{\frac{1}{2}}}{\Psi_z - (-Q/4Q - iQ^{\frac{1}{2}})\,\Psi}\right|_{z_1} \tag{C 4}$$

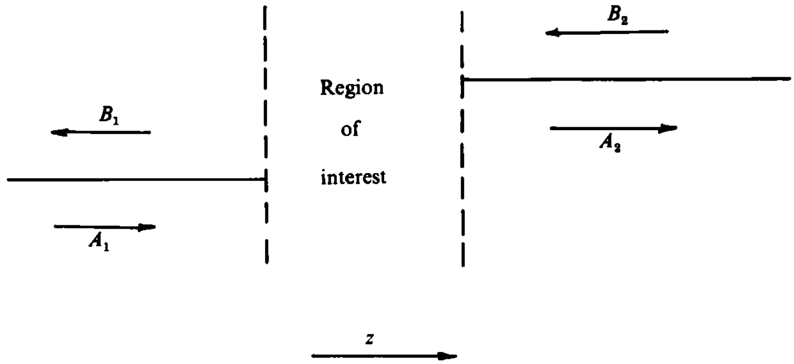


FIGURE 9. Configuration considered in Appendix C.

or, numerically,

$$T = \left| \frac{\psi_{\nu_2}}{\psi_{\nu_1}} \frac{Q^{\frac{1}{2}}_{\nu_1}(\alpha_{\nu_2+1} - \alpha^{-1}_{\nu_2}) - (-Q'_{\nu_2}/4Q_{\nu_2} - \mathrm{i}Q^{\frac{1}{2}}_{\nu_2})}{Q^{\frac{1}{2}}_{\nu_2}(\alpha_{\nu_1+1} - \alpha^{-1}_{\nu_1}) - (-Q'_{\nu_1}/4Q_{\nu_1} - \mathrm{i}Q^{\frac{1}{2}}_{\nu_1})} \right|. \tag{C 5}$$

This is the transmission coefficient of the region $z_1 \leqslant z \leqslant z_2$, provided that there is no downward-moving wave at z_2, i.e. $B_2 = 0$.

In a uniform region, where $Q = m^2 > 0$ and m has the same sign as $U - c$, these formulae reduce to

$$\Psi = A\,\mathrm{e}^{\mathrm{i}mz} + B\,\mathrm{e}^{-\mathrm{i}mz}, \tag{C 6}$$

$$R = \left| \frac{\Psi_z - \mathrm{i}m\Psi}{\Psi_z - \mathrm{i}m\Psi} \right| \simeq \left| \frac{(\alpha_{\nu+1} - \alpha^{-1}_{\psi}) - 2\mathrm{i}m\,\delta z}{(\alpha_{\nu+1} - \alpha^{-1}_{\nu}) - 2\mathrm{i}m\,\delta z} \right|, \tag{C 7}$$

$$T = \left| \frac{m^{\frac{1}{2}}_1[\Psi_z + \mathrm{i}m\Psi]_{z_2}}{m^{\frac{1}{2}}_2[\Psi_z + \mathrm{i}m\Psi]_{z_2}} \right| \approx \left| \frac{m^{\frac{1}{2}}_1\,\Psi_{\nu_2}(\alpha_{\nu_2+1} - \alpha^{-1}_{\nu_2}) + 2\mathrm{i}m\,\delta z}{m^{\frac{1}{2}}_2\,\Psi_{\nu_1}(\alpha_{\psi_1+1} - \alpha^{-1}_{\nu_1}) + 2\mathrm{i}m\,\delta z} \right|. \tag{C 8}$$

In most cases it is impossible to avoid partial reflections off regions where $N^2 \neq$ constant or $U_{zz} \neq 0$, i.e. off inhomogeneities in the index of refraction Q. Then measurement of the reflection and transmission coefficient of a particular region, such as the critical level and its surrounding 'trapping region', is complicated by the fact that there is a small-amplitude wave incident from above as well as the wave from below. Unless Q is symmetric about the critical level, two realizations will be needed to find these coefficients, for example with the principal wave incident from below and then from above.

To see this, consider a region with reflection and transmission coefficients R_1 and T_1 for a wave incident from below, R_2 and T_2 for one from above. Let A_1 and B_1 be the upward and downward waves below, A_2 and B_2 those above (see figure 9). Then, in two realizations (1) and (2),

$$\left.\begin{aligned} B^{(1)}_1 &= A^{(1)}_1 R_1 + B^{(1)}_2 T_2, \quad A^{(1)}_2 = B^{(1)}_2 R_2 + A^{(1)}_1 T_1, \\ B^{(2)}_1 &= A^{(2)}_1 R_1 + B^{(2)}_2 T_2, \quad A^{(2)}_2 = B^{(2)}_2 R_2 + A^{(2)}_1 T_1, \end{aligned}\right\} \tag{C 9}$$

from which R_1, R_2, T_1, T_2 may be found. In the symmetric case, $R_1 = R_2$, $T_1 = T_2$, one set of equations is sufficient. Note that there is no need to worry about the phase in the WKB solution (C 1), provided the same reference points are used in both realizations, in order to get the magnitudes of the Rs and Ts correct.

In the initial-value problem (6.2) the formulae (C 2)–(C 8) may still be applied to the solution $\Psi^{(n)}$ at any time step. However, if there is rapid variation of A and B with z, they may not give very accurate results. In particular, the computed values may show spurious oscillations of wavenumber $2m$, since if $B = B(z)$ in (C 6), for example, (C 7) gives

$$\left|\frac{\Psi_z - \mathrm{i}m\Psi}{\Psi_z - \mathrm{i}m\Psi}\right| = \left|\frac{B}{A}\right| \left|\frac{1+(B'/2mB)}{1+(B'/2mA)\,\mathrm{e}^{-2\mathrm{i}mz}}\right|. \tag{C 10}$$

If such oscillations are observed, it may be advisable to use a different measure for R. We did not find this necessary in any calculation presented in this paper.

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